Energy Transport

[see also Slide in English]

le transport d'énergie peut être vu comme le résultat de 4 mécanismes généraux :

1. Conduction due au mouvement thermique des électrons et des ions
2. Transport radiatif par les photons
3. Convection de paquets macroscopiques de gaz
4. Emission de neutrinos dans les coeurs stellaires.

La conduction et le rayonnement peuvent être vus comme des transports résultant de mouvements aléatoires thermiques des particules.

Dans les étoiles normales, la conduction est négligeable.

le transport radiatif domine généralement tant que le gradient de température reste en dessous d'une valeur critique. Sinon la convection prend le relais

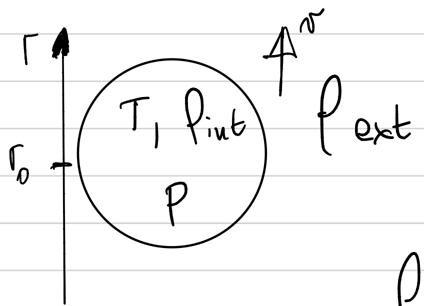
L'émission des neutrinos est importante pour le refroidissement des cœurs d'étoiles massives dans leur stade avancé. C'est un mécanisme de transport d'énergie qui n'opère qu'à haute température et densité. les neutrinos ont peu d'interaction avec l'étoile car ils transportent l'énergie essentiellement à la vitesse de la lumière.

Energy transfer by convection

If the radiative flux is proportional to $\frac{dT}{dr}$, how large can the temperature gradient be?

There is a limit that we will now see

Let's study the motion of a gas cell of volume ΔV, temperature T, pressure P, density ρ_{int}, placed in a medium of density ρ_{ext}



A small petubation puts the cell in motion.

The equation of motion is:

$$\rho_{int} \Delta V \frac{d^2r}{dt^2} = -\rho_{int} \Delta V g + \underbrace{\rho_{ext} \Delta V g}_{\substack{\text{Buyant force} \\ \text{(Archimedes)}}}$$

$$\rho_{int} \frac{d^2r}{dt^2} + (\rho_{int} - \rho_{ext}) g = 0$$

We neglect any possible viscosity

If the petubation is small, the density variation is also small. One can linearize ...

$$\rho_{int}(r) = \rho_{int}(r_0) + \frac{d\rho_{int}}{dr} \underbrace{(r - r_0)}_{\delta r} + \mathcal{O}(r^2)$$

$$\rho_{ext}(r) = \rho_{ext}(r_0) + \frac{d\rho_{ext}}{dr} (r - r_0) + \mathcal{O}(r^2)$$

At equilibrium $\quad P_{int} = P_{ext}$

introducing the above relations in the equation of motion

$$0 = \left[\rho_{int}(r_0) + \frac{d\rho_{int}}{dr}\delta r \right] \frac{d^2 r}{dt^2} + g\left[\rho_{int}(r_0) - \rho_{ext}(r_0) + \frac{d\rho_{int}}{dr}\delta r - \frac{d\rho_{ext}}{dr}\delta r \right]$$

- if r_0 is the position of equilibrium : $\rho_{int}(r_0) = \rho_{ext}(r_0)$
- $\delta r \dfrac{d^2 r}{dt^2}$ is a second order term, hence we can neglect it

il reste

$$\rho_{int}(r_0) \frac{\delta r^2}{dt^2} + g\left(\frac{d\rho_{int}}{dr} - \frac{d\rho_{ext}}{dr} \right) \delta r = 0$$

$$\iff \frac{\delta r^2}{dt^2} + \frac{g}{\rho_{int}}\left(\frac{d\rho_{int}}{dr} - \frac{d\rho_{ext}}{dr} \right) \delta r = 0$$

with $\delta r = r - r_0$

One can recognize the equation of an harmonic oscillator $\equiv \dfrac{d^2 x}{dt^2} + \dfrac{k}{m} x = 0$

of solution $\quad \delta r = r - r_0 = A e^{-iNt}$

with

$$\boxed{N^2 = \frac{g}{\rho}\left(\frac{d\rho_{int}}{dr} - \frac{d\rho_{ext}}{dr} \right) \qquad N = \text{Frequency of Brunt-Väisälä}}$$

It corresponds to the frequency of the g-mode in stars (see slides)

- $N^2 > 0$ if $\dfrac{d\rho_{int}}{dr} > \dfrac{d\rho_{ext}}{dr}$

If $\rho_{int} > \rho_{ext}$, the weight of the element of mass $>$ Archimedes' force

The cell comes back to its initial position $\leftrightarrow$ equilibrium (oscillation)

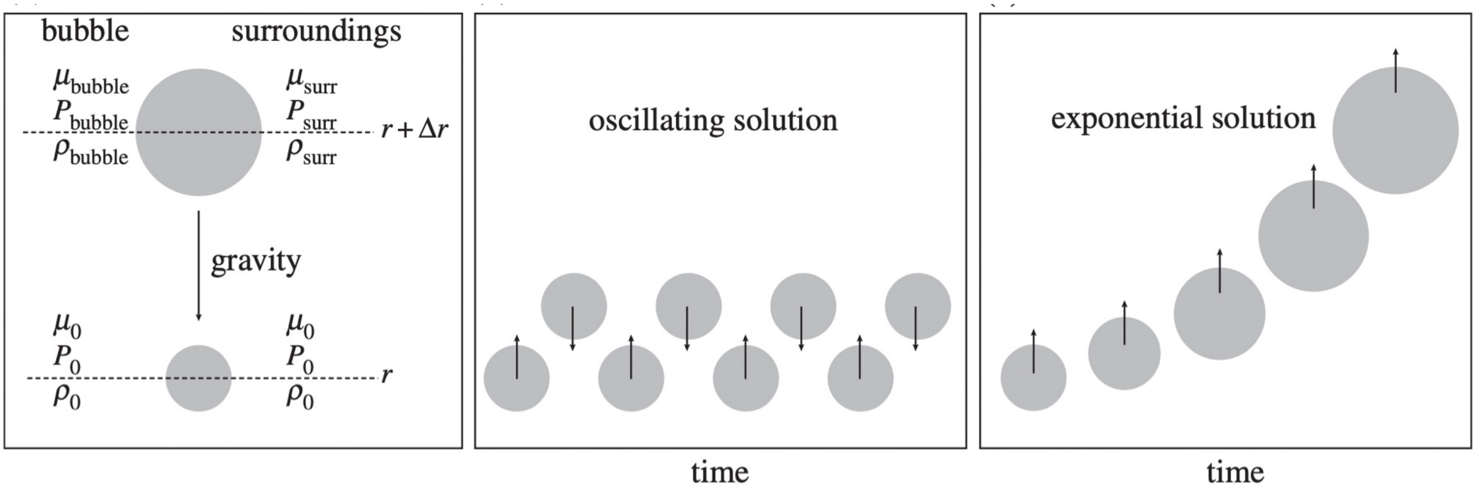
- $N^2 < 0$ if $\dfrac{d\rho_{int}}{dr} < \dfrac{d\rho_{ext}}{dr}$

The cell keeps up moving up

The solution is complex

$$N = \underset{\substack{\uparrow \\ \text{outboard motion}}}{-}\, i\sqrt{\frac{g}{\rho}\left(\frac{d\rho_{ext}}{dr} - \frac{d\rho_{int}}{dr}\right)}$$

Note: The Brunt-Väisälä frequency corresponds to the frequency of a gravity wave, that plays an important role in the energy exchanges of geophysical flows and in oceanography. It controls the position and height of the peaks in the cumulus as well as the swell cust in open sea.



Adiabatic Convection .

- let's assume that the gas cell is sufficiently large to hamper any thermal exchange. (or that this exchange is small enough to be negligible)
- The internal pressure will adjust to the external pressure

Soit C_p = molar specific heat at constant pressure

C_v = molar specific heat at constant volume

$\gamma = \frac{C_p}{C_v}$ and $PV^{\gamma} = cst$ for an ideal gas in adiabatic condition

Taking ρ_{int} , ρ_{ext} as above $\rho_{int} \propto V^{-1}$

$$\Longrightarrow P_{int}\,\rho_{int}^{-\gamma} = cte \Rightarrow \boxed{\rho_{int} \propto P_{int}^{1/\gamma}}$$

We had $$\ddot{\delta r} + \frac{g}{\rho_{int}}\left(\frac{d\rho_{int}}{dr} - \frac{d\rho_{ext}}{dr}\right)\delta r = 0$$

$$\frac{d\rho_{int}}{dr} \propto \frac{d(P_{int}^{1/\gamma})}{dr} = \frac{1}{\gamma} P^{\frac{1}{\gamma}-1}\frac{dP_{int}}{dr} = \frac{1}{\gamma}\frac{\rho_{int}}{P}\frac{dP_{int}}{dr}.$$

At equilibrium around r_0 , $P_{int} \sim P_{ext} \equiv P$:

$\Updownarrow$

$$\frac{d\rho_{int}}{dr} > \frac{d\rho_{ext}}{dr} \quad\longleftrightarrow\quad \boxed{\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} > \frac{1}{\rho}\frac{d\rho}{dr}}$$

$$\underbrace{\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr}}_{\text{Only depends on the gas properties + adiabaticity}} > \underbrace{\frac{1}{\rho}\frac{d\rho}{dr}}_{\text{depends on the star (gravity and hydrostatic equilibrium)}}$$

One can make the link with temperature explicit

Ideal gas $\Rightarrow$ $PV = nRT$

number of moles (n)

universal constant of ideal gases (R)

$$PV \propto T \iff P \propto \rho T$$

$$\frac{dP}{dr} = T\frac{d\rho}{dr} + \rho\frac{dT}{dr}$$

$$\frac{1}{P}\frac{dP}{dr} = \frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{T}\frac{dT}{dr} \Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}$$

The criterion for equilibrium becomes:

$$\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} > \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}$$

$$\frac{1}{P}\frac{dP}{dr}\left(\frac{1}{\gamma} - 1\right) > -\frac{1}{T}\frac{dT}{dr}$$

$$\frac{dT}{T}\frac{P}{dP} < \frac{\gamma - 1}{\gamma} \iff \nabla \equiv \frac{d\ln T}{d\ln P} < \frac{\gamma-1}{\gamma} \equiv \nabla_{ad}$$

Only depends on gas

Schwarzschild criterion

Monoatomic ideal gas: $\gamma = \frac{5}{3}$ $\quad \frac{\gamma - 1}{\gamma} = \frac{2}{5} \sim 0.4$

- if $\frac{dT}{dr}$ small, then no convection
  if $\frac{dT}{dr}$ strong, convection takes place

  as soon as $-\frac{dT}{dr} \gtrsim -\frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr}$

$\left|\frac{dT}{dr}\right|$ stays at its critical value, as indeed convection is a very efficient energy transfer

$\frac{dT}{dr}$ changes at the transition between the radiative and convective zones.

Adiabatic gradient = temperature variation if a gas cell would move up without exchange of heat.

If $\frac{dT}{dr} > \nabla_{ad}$ this gas cell will continue to move up

Convection transfers heat to the surface very rapidly

The cell volume will increase and cool down during its motion

Transfer by electronic conduction

from thermal motion of ions and electrons

Conduction, in the same way as radiation, can be seen as the result of random motions from the particles.

In general, conduction can be neglected. It must be considered in stars composed of degenerated matter (such as white dwarfs), when densities are very high, and the Pauli principle can not be fulfilled any more.

- For an ideal gas, monoatomic = composed of independent atoms, of one element (H, C, etc...)

$$E_{cin} = \frac{3}{2} kT = \frac{1}{2} m v^2$$

ions = $v_i = \sqrt{\dfrac{3kT}{A m_H}}$ $\quad$ A = mass number (protons, neutrons)

electron = $v_e = \sqrt{\dfrac{3kT}{m_e}}$

$$\frac{v_e}{v_i} = \left(\frac{A m_H}{m_e}\right)^{1/2} \simeq 43\sqrt{A} \gg 1$$

$m_H = 1.6726\ 10^{-27}$ kg

$m_e = 9.1094\ 10^{-31}$ kg.

Hence the e^- are transporting the heat

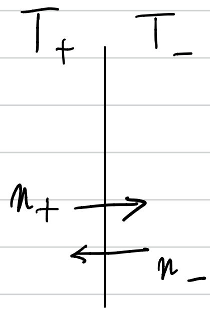
$$n_i = \frac{\rho}{A m_H}$$

$n_e = \frac{1}{2} \frac{\rho}{m_H} (1+X)$; X = hydrogen abundance in mass unit

Let's consider moving e^- and static ions:

$$v = \sqrt{\frac{3kT}{m}} \rightarrow$$ e^- have larger velocities from high temperature to lower ones

n_+ = number of e^- passing from + to −
n_- = − to +



atoms are neutral

hence one must have

$$n_+ = n_-$$

$$E_+ > E_- \Rightarrow n_+ E_+ \neq n_- E_-$$

$\Rightarrow$ There is no flux of particles but there is flux of energy.

At LTE:

$$F_{rad} = -\frac{4acT^3}{3\kappa_{rad}\rho}\frac{dT}{dr} = -C_{rad}\frac{dT}{dr}$$

In the same way, one can write

$$F_{cond} = -C_{cond}\frac{dT}{dr} = -\frac{4acT^3}{3\kappa_{cond}\rho}\frac{dT}{dr}$$

If two modes of transport:

$$F = F_{rad} + F_{cond} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

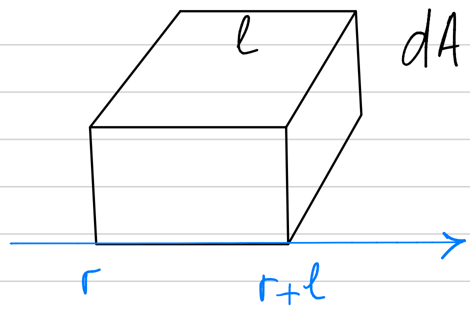
with

$$\frac{1}{\kappa} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}$$

$$[C] = W\,m^{-1}K^{-1} = erg\,/\,s\,/\,cm\,/\,K$$

How to estimate C_{cond} ?

let's follow an intuitive rationale



l = mean free path

E = mean kinetic energy from the e^-

$\overline{v}_e$ = quadratic mean velocity (e^-)

Energy balance : one consider the e^- coming from r and passing through r + l

$$dU = n_e \left[E(r)\, l\, dA - E(r+l)\, l\, dA \right]$$

$$E(r) - E(r+l) \simeq - l \frac{dE}{dr}$$

$$\Rightarrow dU = - n_e\, l^2 \frac{dE}{dr}\, dA \quad \text{and} \quad l = \overline{v}_e\, dt$$

$$F = \frac{dU}{dt\, dA} = - n_e \overline{v}_e\, l \frac{dE}{dr}$$

$$E = \frac{1}{2} m_e \overline{v}_e^2 = \frac{3}{2} kT \Rightarrow \frac{dE}{dr} = \frac{3}{2} k \frac{dT}{dr}$$

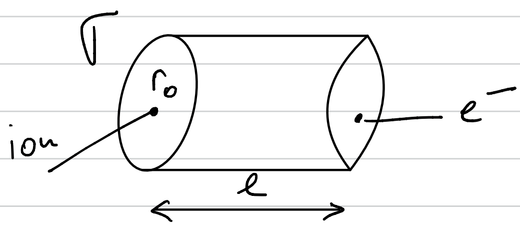
$$F_{cond} = - C_{cond} \frac{dT}{dr} = -\frac{3}{2} n_e \overline{v}_e\, l\, k \frac{dT}{dr} \Rightarrow C_{cond} = \frac{3}{2} n_e k \overline{v}_e\, l$$

n_e, l ?

How much is ℓ ?

For a collision ion, electron

(collisions between e^- are less efficient than i/e^-)



σ = cross-section of the collision

For one collision, one considers a volume V in which there is one ion

$$n_i = \frac{\text{number}}{\text{Volume}}$$

$$1 = n_i V_i = n_i \sigma \ell \Rightarrow \ell = \frac{1}{n_i \sigma}$$

How much is σ ?

Interaction occurs when $E_{kin}(e^-) = E_{pot}(e^-)$

$$\Leftrightarrow \frac{1}{2} m_e \bar{v}_e^2 = \frac{Ze^2}{r_0} \qquad Z = \text{ion atomic number} \quad ; \text{cgs}$$

$$\Rightarrow r_0 = \frac{2 Z e^2}{m_e \bar{v}_e^2} = \frac{2}{3} \frac{Ze^2}{3kT}$$

$$\sigma = \pi r_0^2 = \frac{4\pi Z^2 e^4}{9 (kT)^2}$$

How much is n_e?

$$C_{cond} = \frac{3}{2} k\, n_e\, \bar{v}_e\, \ell \qquad \bar{v}_e = \left(\frac{3kT}{m_e}\right)^{1/2}$$

In the case of full ionisation:

$Z+Y+X=1$

$$n_e = \frac{X\rho}{m_H} + 2\,\frac{Y\rho}{4 m_H} + \frac{\rho}{m_H} \sum_i \frac{X_i Z_i}{A_i}$$

$$\text{et } \frac{Z_i}{A_i} \sim \frac{1}{2}$$

$$n_e = \frac{\rho}{m_H}\left(X + \frac{1}{2}Y + \frac{1}{2}\underbrace{\sum_i X_i}_{Z}\right)$$

$$= \frac{\rho}{m_H}\left(X + \frac{1}{2}\underbrace{(Y+Z)}_{1-X}\right) = \frac{\rho}{2 m_H}(1+X)$$

$$\ell = \frac{1}{n_i \sigma} \; ; \; n_i = \frac{\rho}{A m_H} \; ; \; \sigma = \frac{4\pi Z^2 e^4}{9 (kT)^2}$$

Assembling all information:

$$\boxed{C_{cond} = \frac{27}{16}\,\frac{\sqrt{3}}{\pi}\,\frac{k^{7/2}(1+X)}{m_e^{1/2}\, e^4\, \frac{Z^2}{A}}\, T^{5/2}}$$

On avait $C_{cond} = \dfrac{4acT^3}{3\kappa_{cond}\rho} \Rightarrow \kappa_{cond} \simeq 5000 \dfrac{Z^2/A}{(1+X)\frac{\rho}{10}} T_7^{1/2}$

$T_7 = \dfrac{T}{10^7}$ $\qquad [\kappa_{cond}] = cm^2 g^{-1}$

see slides as well.